Løsningsforslag

Matematik A 14. december 2005

---

Opgave 1

a)

Skitse :

$f(1)=1^2=1 \wedge g(1)=\frac{1}{1}=1$ Hermed vist

b)

$$A_M = \int_0^1 x^2 dx + \int_1^4 \frac{1}{x} dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\ln|x|\right]_1^4 = \left(\frac{1}{3}-0\right)+\left(\ln(4)-0\right) = \underline{\frac{1}{3}+2\ln(2)} \approx 1{,}72$$

c)

$$V_{x,M} = \pi \int_0^1 \left(x^2\right)^2 dx + \pi \int_1^4 \left(\frac{1}{x}\right)^2 dx = \pi \int_0^1 x^4 dx + \pi \int_1^4 \frac{1}{x^2} dx = \pi \left[\frac{x^5}{5}\right]_0^1 + \pi \left[-\frac{1}{x}\right]_1^4 =$$

$$\pi\left(\left(\frac{1}{5}-0\right)+\left(-\frac{1}{4}+1\right)\right) = \underline{\frac{19}{20}\pi} \approx 2{,}98$$

d)

$$V_{y,M} = 2\pi \int_0^1 x \cdot x^2 dx + 2\pi \int_1^4 x \cdot \frac{1}{x} dx = 2\pi \int_0^1 x^3 dx + 2\pi \int_1^4 dx = \quad 2\pi \left[\frac{x^4}{4}\right]_0^1 + 2\pi \left[x\right]_1^4 =$$

$$2\pi\left(\frac{1}{4}-0+4-1\right) = \underline{\frac{13}{2}\pi} \approx 20{,}4$$

Opgave 2

a)

$$\frac{dy}{dx}=\frac{\cos(x)}{2y} \quad , \quad x\in \mathrm{R} \ , \ y>0 \Leftrightarrow$$

$$\int 2y dy=\int \cos(x)dx+k_1 \ , \ k_1\in R \ , \ y>0 \Leftrightarrow$$

$$y^2=\sin(x)+k_1 \ , \ k_1\in R \ , \ y>0 \Leftrightarrow$$

$$\underline{y=\sqrt{\sin(x)+k_1}} \quad , \ k_1\in R \ , \ \sin(x)+k_1>0$$

b)

$$\alpha_t=f'(0)=\frac{dy}{dx}(0,3)=e^{2\cdot 0}+3\cdot 3=1+9=10$$

$$y-f(0)=\alpha_t(x-0)\Leftrightarrow y-3=10(x-0)\Leftrightarrow \underline{y=10x+3}$$

c)

$$g(x)=e^x\sin(x)\Rightarrow g'(x)=e^x\sin(x)+e^x\cos(x)$$

$$\frac{dy}{dx}=e^x\cos(x)+y\Rightarrow e^x\sin(x)+e^x\cos(x)=e^x\cos(x)+e^x\sin(x)\Leftrightarrow 0=0$$

Hermed vist

Opgave 3

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sqrt{t}-2 \\ \ln(t) \end{pmatrix},\ t>0\ ;\ \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{t}} \\ \frac{1}{t} \end{pmatrix},\ t>0\ ;\ \begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} -\frac{1}{4t\sqrt{t}} \\ -\frac{1}{t^2} \end{pmatrix},\ t>0$$

a)

$$x=0 \Leftrightarrow \sqrt{t}-2=0 \Leftrightarrow \sqrt{t}=2 \Leftrightarrow t=4 \Rightarrow \underline{(x,y)=(0,2\ln(2))} \approx (0;1,39)$$

$$y=0 \Leftrightarrow \ln(t)=0 \Leftrightarrow t=1 \Rightarrow \underline{(x,y)=(-1,0)}$$

b)

$$x'=0 \Leftrightarrow \frac{1}{2\sqrt{t}}=0 \Leftrightarrow 1=0 \Leftrightarrow t\in\emptyset$$

$$y'=0 \Leftrightarrow \frac{1}{t}=0 \Leftrightarrow 1=0 \Leftrightarrow t\in\emptyset$$

Hermed vist

c)

$$\begin{pmatrix} x(1) \\ y(1) \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}\ ;\ \begin{pmatrix} x'(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \Rightarrow \alpha_t = \frac{y'(1)}{x'(1)} = \frac{1}{1/2} = 2$$

$$y-y(1)=\alpha_t(x-x(1)) \Leftrightarrow y-0=2(x-(-1)) \Leftrightarrow \underline{y=2x+2}$$

d)

$$\begin{pmatrix} x'(2) \\ y'(2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}\ ;\ \begin{pmatrix} x'(2) \\ y'(2) \end{pmatrix} = \begin{pmatrix} -\frac{1}{8\sqrt{2}} \\ -\frac{1}{4} \end{pmatrix}$$

$$\cos\angle\left(\vec{v}(2),\vec{a}(2)\right)=\frac{\begin{pmatrix}\frac{1}{2\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{8\sqrt{2}}\\ -\frac{1}{4}\end{pmatrix}}{\left|\begin{pmatrix}\frac{1}{2\sqrt{2}}\\ \frac{1}{2}\end{pmatrix}\right|\left|\begin{pmatrix}\frac{1}{8\sqrt{2}}\\ -\frac{1}{4}\end{pmatrix}\right|}=\frac{\frac{1}{2\sqrt{2}}\cdot\left(-\frac{1}{8\sqrt{2}}\right)+\frac{1}{2}\cdot\left(-\frac{1}{4}\right)}{\sqrt{\frac{1}{8}+\frac{1}{4}}\sqrt{\frac{1}{128}+\frac{1}{16}}}=\frac{-\frac{1}{32}-\frac{1}{8}}{\sqrt{\frac{3}{8}}\sqrt{\frac{9}{128}}}=$$

$$\frac{-\frac{5}{32}}{\frac{3\sqrt{3}}{32}}=\frac{-5}{3\sqrt{3}}\approx-0{,}9623\Rightarrow \underline{\angle\left(\vec{v}(2),\vec{a}(2)\right)\approx 164{,}21^\circ}$$ (Spids vinkel : 15,79°)

e)

$$\begin{pmatrix}x'\\ y'\end{pmatrix}\square\begin{pmatrix}1\\ 1\end{pmatrix}\Leftrightarrow\begin{pmatrix}\frac{1}{2\sqrt{t}}\\ \frac{1}{t}\end{pmatrix}\cdot\begin{pmatrix}-1\\ 1\end{pmatrix}=0\Leftrightarrow-\frac{1}{2\sqrt{t}}+\frac{1}{t}=0\Leftrightarrow\frac{1}{t}=\frac{1}{2\sqrt{t}}\Leftrightarrow 2\sqrt{t}=t\Leftrightarrow$$

$$2\sqrt{t}-t=0\Leftrightarrow\sqrt{t}\left(2-\sqrt{t}\right)=0\Leftrightarrow t=0\vee\sqrt{t}=2\Leftrightarrow t=0\vee t=4\quad(t>0)$$

Den søgte tangent findes altså for $t=4$ :

$$\begin{pmatrix}x(4)\\ y(4)\end{pmatrix}=\begin{pmatrix}0\\ 2\ln(2)\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x(4)\\ y(4)\end{pmatrix}+s\begin{pmatrix}1\\ 1\end{pmatrix}\Leftrightarrow\underline{\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 2\ln(2)\end{pmatrix}+s\begin{pmatrix}1\\ 1\end{pmatrix}\ ,\ s\in\mathrm{R}}$$

Opgave 4

a)

$$\overrightarrow{AB}=\begin{pmatrix}1-0\\-2-2\\1+1\end{pmatrix}=\begin{pmatrix}1\\-4\\2\end{pmatrix}\ ;\ \overrightarrow{AC}=\begin{pmatrix}-1-0\\0-2\\3+1\end{pmatrix}=\begin{pmatrix}-1\\-2\\4\end{pmatrix}$$

$$\vec{n}_\alpha=\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}\begin{vmatrix}-4&-2\\2&4\end{vmatrix}\\\begin{vmatrix}2&4\\1&-1\end{vmatrix}\\\begin{vmatrix}1&-1\\-4&-2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}-16+4\\-2-4\\-2-4\end{pmatrix}=\begin{pmatrix}-12\\-6\\-6\end{pmatrix}=-6\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$2(x-0)+1(y-2)+1(z+1)=0\Leftrightarrow 2x+y-2+z+1=0\Leftrightarrow \underline{2x+y+z-1=0}$$

$$\vec{r}_m=\overrightarrow{PQ}=\begin{pmatrix}-2-3\\1+2\\1-0\end{pmatrix}=\begin{pmatrix}-5\\3\\1\end{pmatrix}\Rightarrow \underline{\mathrm{m}\ :\ \begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-2\\0\end{pmatrix}+t\begin{pmatrix}-5\\3\\1\end{pmatrix}\ ,\ t\in R}$$

b)

$$x=3-5t\wedge y=-2+3t\wedge z=t\wedge 2x+y+z-1=0\Leftrightarrow 6-10t-2+3t+t-1=0\Leftrightarrow$$

$$3=6t\Leftrightarrow t=\frac{1}{2}\Rightarrow S\left(3+\frac{1}{2}\cdot(-5),-2+\frac{1}{2}\cdot 3,0+\frac{1}{2}\cdot 1\right)\Leftrightarrow \underline{S\left(\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)}$$

c)

$$\vec{r}_n=\overrightarrow{OA}=\begin{pmatrix}0\\2\\-1\end{pmatrix}\ ;\ \vec{n}=\vec{r}_m\times\vec{r}_n=\begin{pmatrix}\begin{vmatrix}3&2\\1&-1\end{vmatrix}\\\begin{vmatrix}1&-1\\-5&0\end{vmatrix}\\\begin{vmatrix}-5&0\\3&2\end{vmatrix}\end{pmatrix}=\begin{pmatrix}-3-2\\0-5\\-10-0\end{pmatrix}=\begin{pmatrix}-5\\-5\\-10\end{pmatrix}=-5\begin{pmatrix}1\\1\\2\end{pmatrix}$$

$$\left|\vec{n}\right|=\sqrt{1^2+1^2+2^2}=\sqrt{6}\ ;\ \vec{n}\cdot\overrightarrow{OP}=1\cdot 3+1\cdot(-2)+2\cdot 0=3-2+0=1$$

c) fortsat

$$dist(m,n) = \frac{\left|\vec{n} \cdot \overrightarrow{OP}\right|}{\left|\vec{n}\right|} = \underline{\frac{1}{\sqrt{6}}}$$

d)

$$T = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\left|\vec{n}_\alpha\right| = \frac{1}{2}\left|-6\right|\sqrt{2^2+1^2+1^2} = \underline{3\sqrt{6}}$$

e) $10{,}52^\circ$

Opgave 5

a)

$$\int_1^9\left(\frac{2}{\sqrt{x}}-\sqrt{x}\right)dx=\int_1^9\left(2x^{-1/2}-x^{1/2}\right)dx=\left[2\cdot 2x^{1/2}-\frac{2}{3}x^{3/2}\right]_1^9=4\cdot 3-\frac{2}{3}\cdot 9\cdot 3-\left(4-\frac{2}{3}\right)=$$

$$12-18-4+\frac{2}{3}=\underline{-\frac{28}{3}}\approx -9{,}33$$

b)

$$\int x^{-1/2}\ln(x)\,dx=2x^{1/2}\ln(x)-\int\frac{1}{x}\cdot 2x^{1/2}dx=2\sqrt{x}\ln(x)-\int 2x^{-1/2}dx=$$

$$2\sqrt{x}\ln(x)-4x^{1/2}+k=\underline{2\sqrt{x}\ln(x)-4\sqrt{x}+k\ ,\ k\in R}$$

idet $\left(\ln(x)\right)'=\frac{1}{x}$ og $\int x^{-1/2}dx=2x^{1/2}$.

c)

$$\int\frac{1}{2\sqrt{x}}\left(1+\sqrt{x}\right)^{-3}dx=\int t^{-3}dt=-\frac{1}{2}t^{-2}+k=-\frac{1}{2}\left(1+\sqrt{x}\right)^{-2}+k=$$

$$\underline{-\frac{1}{2\left(1+\sqrt{x}\right)^2}+k\ ,\ k\in R}$$

idet $t=1+\sqrt{x}\Rightarrow\frac{dt}{dx}=\frac{1}{2\sqrt{x}}\Leftrightarrow dt=\frac{1}{2\sqrt{x}}dx$.