Brush up-kursus

Matematik

Juni 2008

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Aritmetik

Addition, subtraktion og parenteser

Den kommutative lov

$$a+b=b+a$$

Den associative lov

$$a+(b+c)=(a+b)+c$$

Multiplikation

Den kommutativ lov

$$a\cdot b=b\cdot a$$

Den associative lov

$$a\cdot(b\cdot c)=(a\cdot b)\cdot c$$

Den distributive lov

$$a\cdot(b+c)=ab+a\cdot c$$

Kvadratet på en toleddet størrelse

$$(a+b)^2=a^2+b^2+2ab$$

$$(a-b)^2=a^2+b^2-2ab$$

To tals sum gange de samme to tals differens

$$(a+b)\cdot(a-b)=a^2-b^2$$

Division og brøker

Brøkregneregler

$$\frac{a}{b} \cdot m = \frac{a \cdot m}{b} \qquad \wedge\ b \neq 0$$

$$\frac{\frac{a}{b}}{m} = \frac{a}{b \cdot m} \qquad \wedge\ b \neq 0 \wedge m \neq 0$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a \cdot d}{b \cdot c} \qquad \wedge\ b \neq 0 \wedge d \neq 0 \wedge c \neq 0$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} \qquad \wedge\ b \neq 0 \wedge d \neq 0$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + bc}{b \cdot d} \qquad \wedge\ b \neq 0 \wedge d \neq 0$$

Rod og potens

Potensregneregler

$$\sqrt[q]{x^p} = x^{\frac{p}{q}}$$

$$x^n \cdot x^m = x^{n+m}$$

$$\frac{x^n}{x^m} = x^{n-m} \qquad \wedge\ x \neq 0$$

$$(x^n)^m = x^{n \cdot m}$$

$$\frac{x^n}{y^n} = \left(\frac{x}{y}\right)^n \qquad \wedge\ y \neq 0$$

$$x^n \cdot y^n = (x \cdot y)^n$$

Eksempel 1

Reducer følgende udtryk

$$(a+2)^2 - (a+2)(a-2) - (a-2)^2$$

$$(a+2)^2 - (a+2)(a-2) - (a-2)^2 =$$
$$a^2+4+4a - (a^2-4) - (a^2+4-4a) =$$
$$a^2+4+4a - a^2+4 - a^2-4+4a =$$
$$\underline{\underline{-a^2+8a+4}}$$

Eksempel 2

Reducer følgende udtryk

$$\frac{1}{a-b} - \frac{b}{a^2-b^2}$$

$$\frac{1}{a-b} - \frac{b}{a^2-b^2} = \frac{1}{a-b} - \frac{b}{(a+b)(a-b)} =$$

$$\frac{a+b}{(a-b)(a+b)} - \frac{b}{(a+b)(a-b)} = \frac{a+b-b}{a^2-b^2} = \underline{\underline{\frac{a}{a^2-b^2}}}$$

Eksempel 3

Reducer følgende udtryk $\left(\frac{14x^5y^8}{7x\,y^{13}}\right)^3$

$$\left(\frac{14x^5y^8}{7x\,y^{13}}\right)^3 = \left(\frac{2x^4}{y^5}\right)^3 = \underline{\underline{\frac{8x^{12}}{y^{15}}}} = 8x^{12}\cdot y^{-15}$$

Eksempel 4

Reducer følgende udtryk $\left(\frac{y^4 \cdot x^{\frac{2}{3}}}{\sqrt[3]{y^5} \cdot \sqrt{x^3}} \cdot x^0\right)^5$

$$\left(\frac{y^4 \cdot x^{\frac{2}{3}}}{\sqrt[3]{y^5} \cdot \sqrt{x^3}} \cdot x^0\right)^5 = \left(\frac{y^4 \cdot x^{\frac{2}{3}} \cdot 1}{y^{\frac{5}{3}} \cdot x^{\frac{3}{2}}}\right)^5 =$$

$$\left(y^{4-\frac{5}{3}} \cdot x^{\frac{2}{3}-\frac{3}{2}}\right)^5 = \left(y^{\frac{7}{3}} \cdot x^{-\frac{5}{6}}\right)^5 = \underline{\underline{y^{\frac{35}{3}} \cdot x^{-\frac{25}{6}}}}$$

Algebra

Ligninger og regneregler

$$a = b \Rightarrow a + c = b + c$$
$$a = b \Rightarrow a - c = b - c$$
$$a = b \Rightarrow ac = bc$$
$$a = b \Rightarrow \frac{a}{c} = \frac{b}{c} \quad \wedge \quad c \neq 0$$

Nulreglen

$$a \cdot b = 0 \Rightarrow a = 0 \vee b = 0$$

Grundmængde

Grundmængden for en ligning er den mængde af tal, der kan indsættes i ligningen.

Eksempel 5

Bestem grundmængden og løsningsmængden til følgende ligning

$$\frac{1}{\ln x} = 4 \Rightarrow$$

$$\underline{\underline{G = \mathbb{R}_+ \setminus \{1\}}}$$

$$4 \ln x = 1 \Rightarrow \ln x = \frac{1}{4} \Rightarrow x = e^{\frac{1}{4}} \Rightarrow$$

$$\underline{\underline{L_x = \{e^{\frac{1}{4}}\}}}$$

$$L_x \subseteq G$$

Eksempel 6

Bestem grundmængden og løsnings-mængden til følgende ligning

$$\frac{1}{2x+3} - \frac{1+x}{4x+6} + \frac{2}{6x+9} = 1 \Rightarrow$$

$$\underline{\underline{G = \mathbb{R} \setminus \{-\tfrac{3}{2}\}}}$$

$$\frac{6}{12x+18} - \frac{3+3x}{12x+18} + \frac{4}{12x+18} = 1 \Rightarrow$$

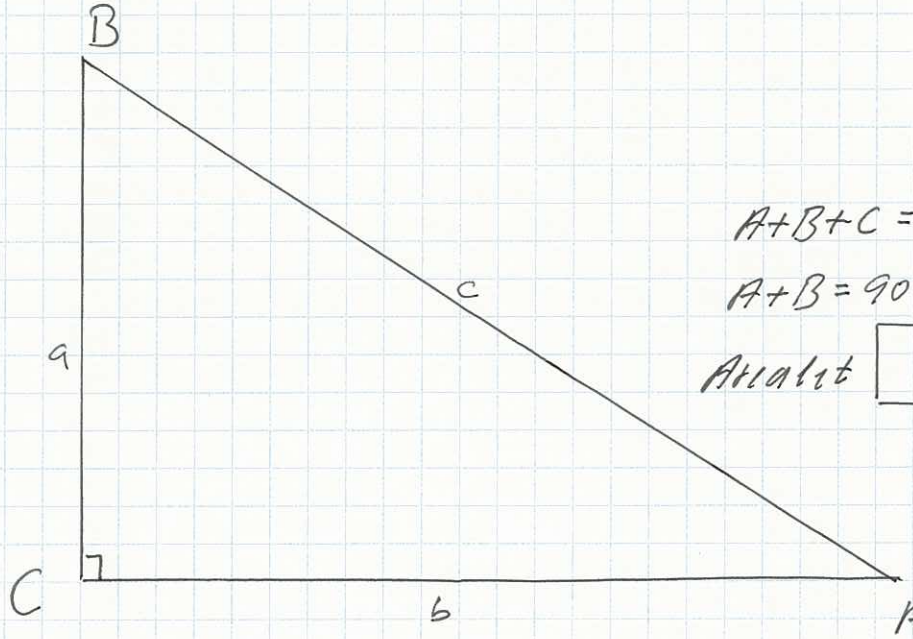
$$6 - 3 - 3x + 4 = 12x + 18 \Rightarrow$$

$$15x = -11 \Rightarrow \quad x = \frac{-11}{15} \Rightarrow$$

$$\underline{\underline{L_x = \{\tfrac{-11}{15}\}}} \qquad L_x \subseteq G$$

Geometri

Den retvinklede trekant



$A+B+C = 180^\circ \Rightarrow$

$A+B = 90^\circ$

Arealet $\boxed{T = \frac{1}{2}ab}$

Den pythagoræiske læresætning

$$\boxed{c^2 = a^2 + b^2}$$

Sinus, cosinus og tangens

$$\sin A = \frac{a}{c} \qquad \sin B = \frac{b}{c} = \cos A$$

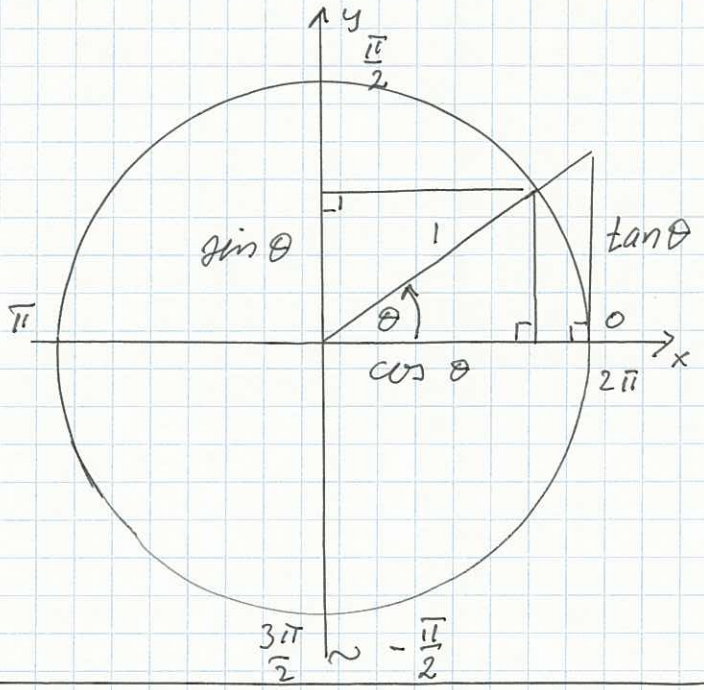
$$\cos A = \frac{b}{c} \qquad \cos B = \frac{a}{c} = \sin A$$

$$\tan A = \frac{a}{b} \qquad \tan B = \frac{b}{a} = \frac{1}{\tan A}$$

$$\tan A = \frac{\sin A}{\cos A}$$

$$\tan B = \frac{\sin B}{\cos B}$$

Enhedscirklen



$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \quad 1 + \tan^2\theta = \frac{1}{\cos^2\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

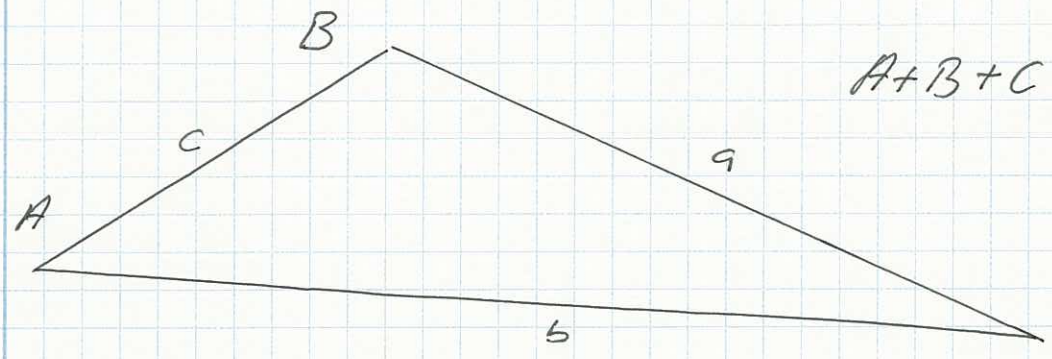
$$\cos\theta \geq 0 \Rightarrow \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\cos\theta < 0 \Rightarrow \theta \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$$

$$\sin\theta \geq 0 \Rightarrow \theta \in [0, \pi]$$

$$\sin\theta < 0 \Rightarrow \theta \in]\pi, 2\pi[$$

Den vilkårlige trekant



$A+B+C=180^\circ$

Arealet $\boxed{T=\frac{1}{2}ab\sin C=\frac{1}{2}ac\sin B=\frac{1}{2}bc\sin A}$

Cosinusrelationerne

$$a^2=b^2+c^2-2bc\cos A$$
$$b^2=a^2+c^2-2ac\cos B$$
$$c^2=a^2+b^2-2ab\cos C$$
$$\cos A=\frac{b^2+c^2-a^2}{2bc}$$
$$\cos B=\frac{a^2+c^2-b^2}{2ac}$$
$$\cos C=\frac{a^2+b^2-c^2}{2ab}$$

Sinusrelationerne

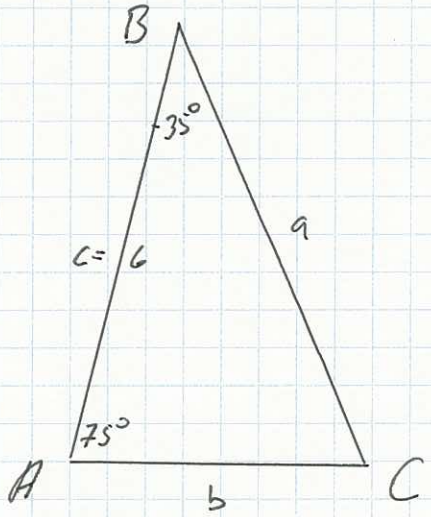
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$
$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

Eksempel 7

I trekant ABC er givet: $A = 75°$, $B = 35°$ og $c = 6$

Beregn: C, a, b og T.

$C = 180 - 75 - 35 = \underline{\underline{70°}}$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \Rightarrow$$

$$a = \frac{c \sin A}{\sin C} = \frac{6 \sin 75°}{\sin 70°} \approx \underline{\underline{6{,}17}}$$

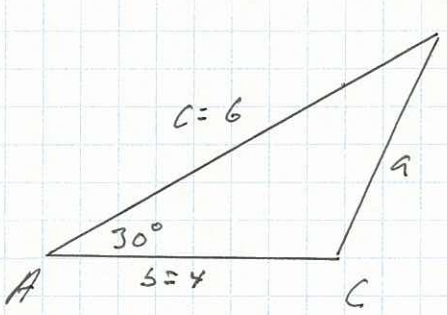
$$b = \frac{c \sin B}{\sin C} = \frac{6 \sin 35°}{\sin 70°} \approx \underline{\underline{3{,}66}}$$

$$T = \tfrac{1}{2} bc \sin A = \tfrac{1}{2} \cdot 3{,}66 \cdot 6 \sin 75° \approx \underline{\underline{10{,}61}}$$

Eksempel 8

I trekant ABC er givet: $A = 30°$, $c = 6$ og $b = 4$

Beregn: a.



$$a^2 = b^2 + c^2 - 2bc \cos A$$

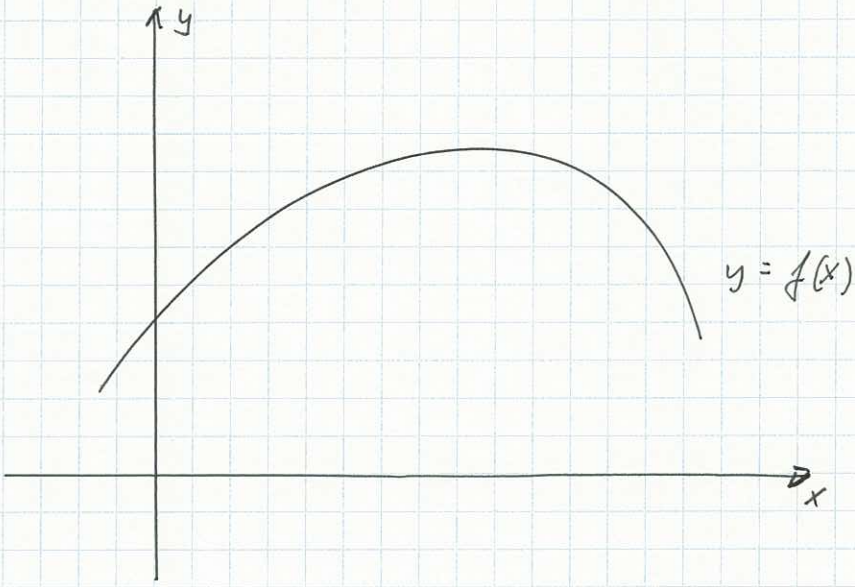
$$a = \sqrt{16 + 36 - 2 \cdot 4 \cdot 6 \cos 30°} \Rightarrow$$

$$\underline{\underline{a \approx 3{,}23}}$$

Funktioner

$y = f(x)$ udtrykker at y er en funktion af x.

x er den uafhængige variabel og y er den afhængige variabel.



Definitionsmængde D_f

D_f er mængden af de x, der kan indsættes i $f(x)$.

Værdimængde V_f

V_f er mængden af de y der optræder når alle x tilhørende D_f indsættes i $f(x)$.

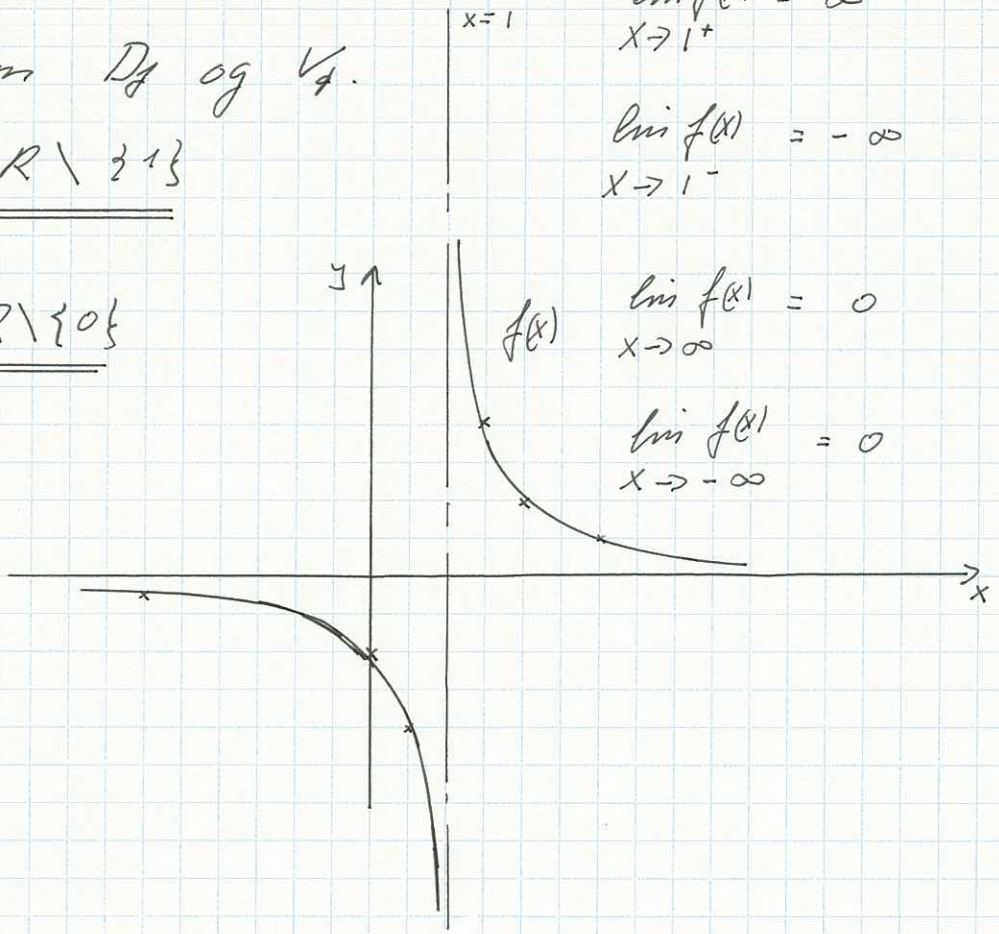
Eksempel 9

$$y = f(x) = \frac{1}{x-1}$$

Bestem D_f og V_f.

$D_f = R \setminus \{1\}$

$V_f = R \setminus \{0\}$



$\lim_{x \to 1^+} f(x) = \infty$

$\lim_{x \to 1^-} f(x) = -\infty$

$\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$

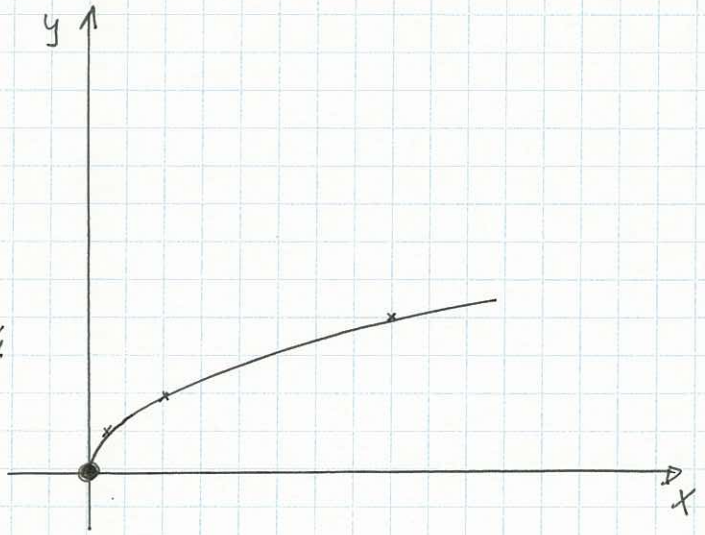
Eksempel 10

$$y = f(x) = \sqrt{x}$$

Bestem D_f og V_f

$D_f = [0, \infty[$

$V_f = [0, \infty[$



Sammensatte funktioner

$$y = f(g(x)) = f \circ g$$

For at kunne danne den sammensatte funktion $f(g(x))$, skal værdimængden af g være en delmængde af definitionsmængden af f.

$$\boxed{V_g \subseteq D_f}$$

Eksempel 11

Givet $f(x) = \sqrt{x}$ og $g(x) = e^x$

Bestem $f(g(x))$ og $D_{f \circ g}$.

$D_f = [0, \infty[$ $\quad$ $D_g = R$

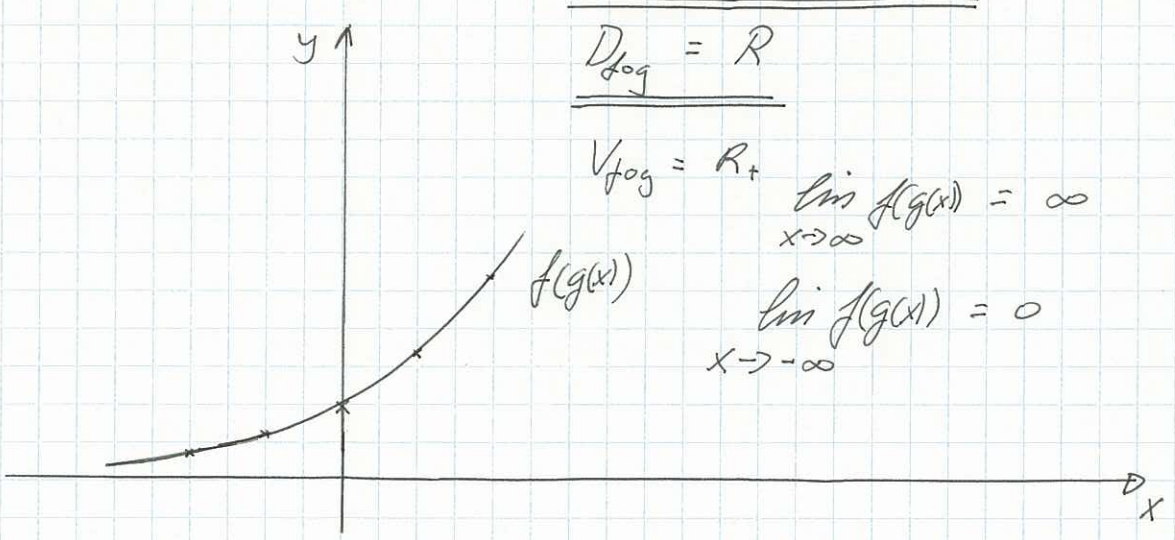
$V_f = [0, \infty[$ $\quad$ $V_g = R_+$

$V_g \subseteq D_f \Rightarrow \underline{\underline{f(g(x)) = \sqrt{e^x}}}$

$\underline{\underline{D_{f \circ g} = R}}$

$V_{f \circ g} = R_+$ $\quad \lim_{x \to \infty} f(g(x)) = \infty$

$\lim_{x \to -\infty} f(g(x)) = 0$



Logaritme- og eksponentialfunktioner

Den naturlige logaritmefunktion

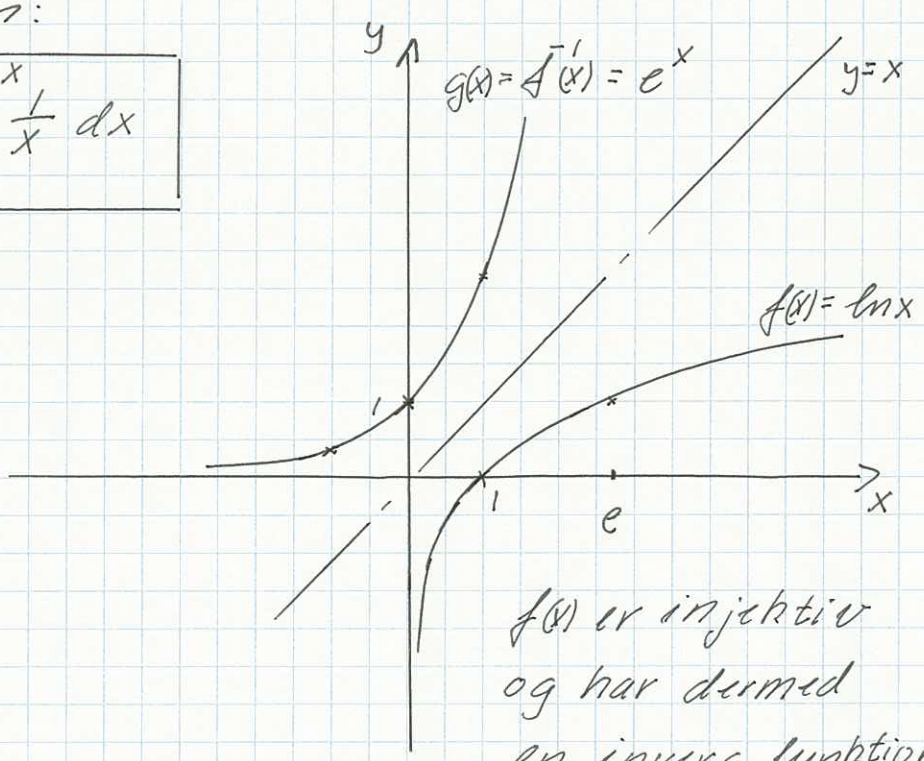
Definition:

$$\ln x = \int_1^x \frac{1}{x}\,dx$$

$f(x) = \ln x$

$D_f = R_+$

$V_f = R$



$f(x)$ er injektiv og har dermed en invers funktion.

eksponentialfunktionen med grundtallet e

Definition:

e^x er den inverse funktion til $\ln x$

$g(x) = f^{-1}(x) = e^x$

$D_g = V_f = R$

$V_g = D_f = R_+$

eksponentialfunktionen med grundtallet a

Definition: $a^x = e^{x \ln a}$ $\wedge$ $a > 0$

Eksponentielle og logaritmiske ligninger

Potensregneregler se side 2
Logaritmeregneregler

$$\ln(a \cdot b) = \ln a + \ln b$$
$$\ln \frac{a}{b} = \ln a - \ln b$$
$$\ln a^b = b \ln a$$

Eksempel 12
Løs ligningen $4^{x-1} - 2 \cdot 4^{1-x} = 3{,}5$

$G = R$ $\quad 4^{x-1} - 2 \cdot 4^{1-x} = 3{,}5 \Rightarrow$

$$4^x \cdot 4^{-1} - 2 \cdot 4 \cdot 4^{-x} = 3{,}5 \Rightarrow$$

$$\frac{4^x}{4} - \frac{8}{4^x} = 3{,}5 \Rightarrow \frac{1}{4} 4^{2x} - 8 - 3{,}5 \cdot 4^x = 0$$

$$\Rightarrow 4^{2x} - 14 \cdot 4^x - 32 = 0 \Rightarrow$$

$$4^x = 7 \pm \sqrt{49 + 32} = \begin{cases} 16 \\ -2 \text{ falsk rod} \end{cases} \Rightarrow$$

$$4^x = 16 \Rightarrow x \ln 4 = \ln 16 \Rightarrow$$

$$x = \frac{\ln 16}{\ln 4} = \frac{2 \ln 4}{\ln 4} = 2 \Rightarrow \underline{\underline{L_x = \{2\}}}$$

Eksempel 13

Løs ligningen $\ln(x+7) - \ln(x-1) = 1$

$G =]1, \infty[$

$$\ln(x+7) - \ln(x-1) = 1 \Rightarrow$$

$$\ln \frac{x+7}{x-1} = 1 \Rightarrow$$

$$\frac{x+7}{x-1} = e \Rightarrow$$

$$x+7 = ex - e \Rightarrow$$

$$x - ex = -e - 7 \Rightarrow$$

$$x(1-e) = -e - 7 \Rightarrow$$

$$x = \frac{e+7}{e-1} \approx 5{,}66 \Rightarrow L_x = \left\{ \frac{e+7}{e-1} \right\}$$

Eksempel 14

Løs ligningen $\ln x^2 - \ln x^3 + 4\ln x - 6 = 0$

$G = \mathbb{R}_+$

$$\ln x^2 - \ln x^3 + 4\ln x - 6 = 0 \Rightarrow$$

$$2\ln x - 3\ln x + 4\ln x = 6 \Rightarrow$$

$$3\ln x = 6 \Rightarrow$$

$$\ln x = 2 \Rightarrow$$

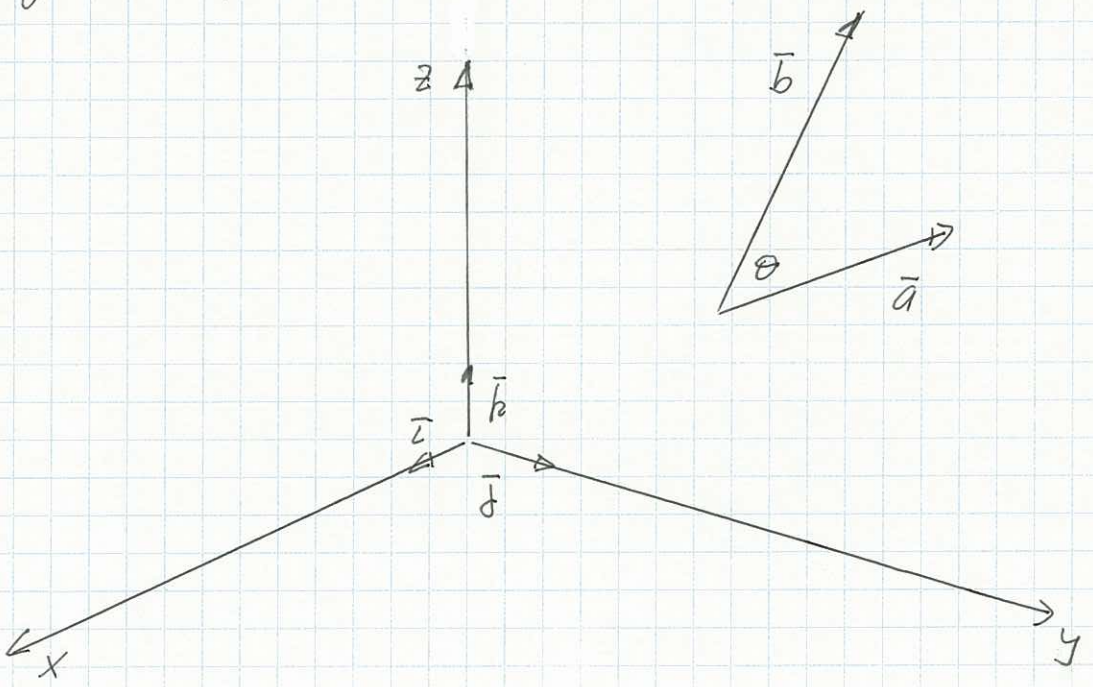
$$x = e^2 \Rightarrow L_x = \{e^2\}$$

Vektorregning

Skalarprodukt (prikprodukt)

$$\bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cos\theta$$

Den kommutative lov gælder for skalarproduktet. $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$



$\bar{a} = (a_1, a_2, a_3)$ $\bar{b} = (b_1, b_2, b_3)$

$$\bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$\bar{a} \cdot \bar{b} = 0 \iff \bar{a} \perp \bar{b}$$

$|\bar{a}|$ længden af $\bar{a}$

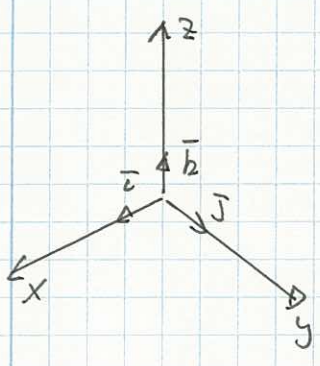
$|\bar{b}|$ længden af $\bar{b}$

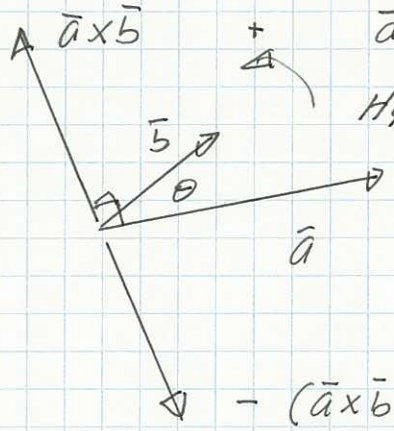
$$|\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\bar{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

Vektorprodukt (krydsprodukt)

$$|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}| \sin\theta$$





$\bar{a}\times\bar{b} \perp \bar{a}$ ∧
$\bar{a}\times\bar{b} \perp \bar{b}$

Højreskrueregten

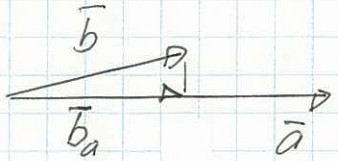
Vektorproduktet følger ikke den kommutative lov.

$$\bar{a}\times\bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) =$$

$$(a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1)$$

$$\bar{a}\times\bar{b} = \bar{0} \iff \bar{a} \parallel \bar{b}$$

Projektion af vektor på vektor



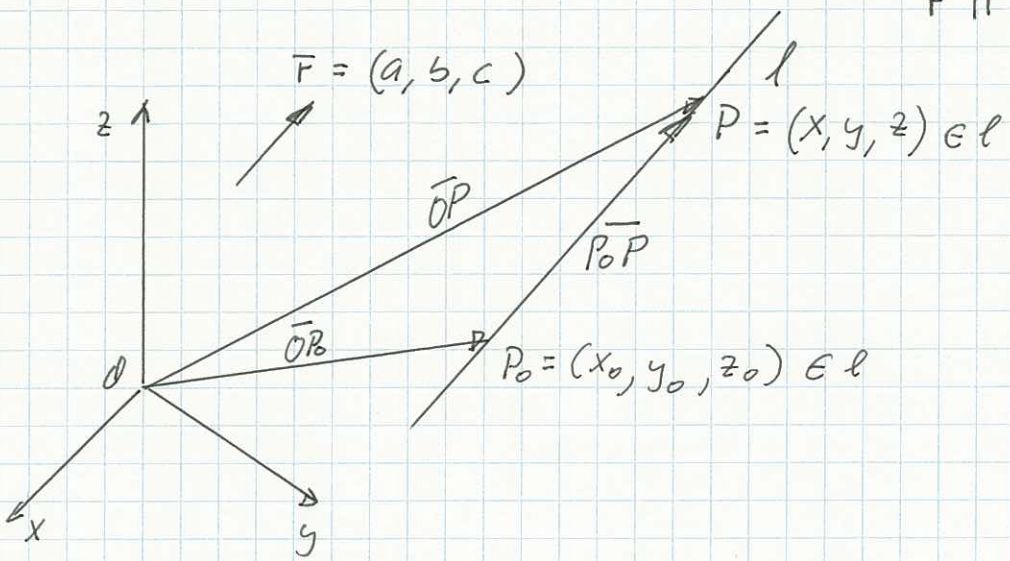
$$\bar{b}_a = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|^2}\cdot\bar{a}$$

$$\bar{a}_b = \frac{\bar{a}\cdot\bar{b}}{|\bar{b}|^2}\cdot\bar{b}$$

Linjer i rummet

parameterfremstilling

$\bar{r} \parallel l$



$\bar{OP}$ er stedvektor til P

$\bar{OP_0}$ er stedvektor til P_0

$\bar{r}$ er retningsvektor til l

$$\bar{OP} = \bar{OP_0} + \bar{P_0P} = \bar{OP_0} + t\bar{r} \quad \wedge \; t \in R$$

$$l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \wedge \; t \in R$$

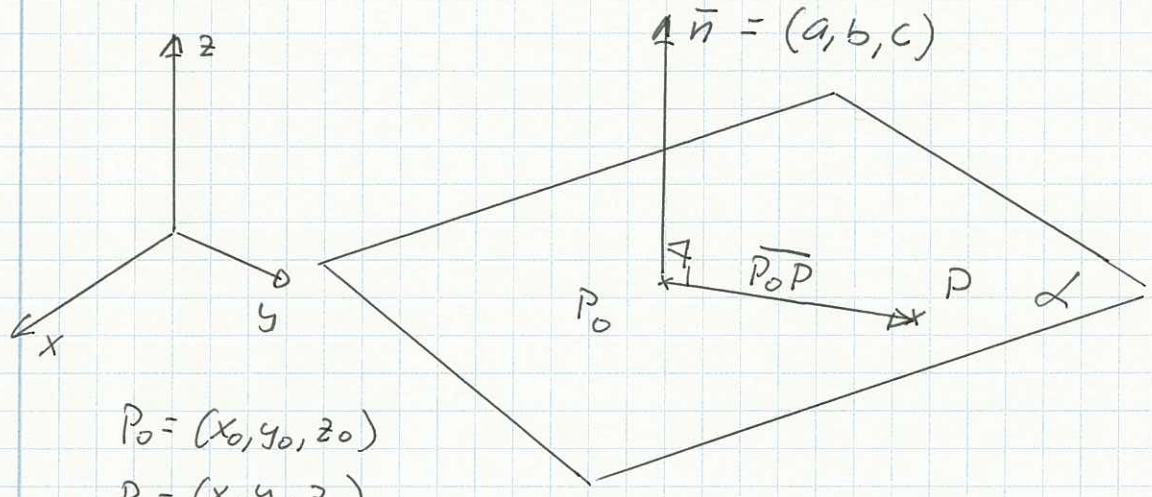
t kaldes parameteren

Eksempel 15

Bestem parameterfremstillingen for den linje l, der går gennem punktet $P_0 = (1, 2, -3)$ og er parallel med $\bar{r} = (-4, 1, 5)$

$$l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t \begin{pmatrix} -4 \\ 1 \\ 5 \end{pmatrix} \quad \wedge \; t \in R$$

Planer



$P_0 = (x_0, y_0, z_0)$

$P = (x, y, z)$

$P_0 \in \alpha \wedge P \in \alpha \Rightarrow \overline{P_0P} \in \alpha$

$\overline{P_0P} = (x - x_0,\ y - y_0,\ z - z_0)$

$\bar{n}$ er normalvektor til α

$\bar{n} \perp \alpha \Rightarrow \bar{n} \perp \overline{P_0P} \Rightarrow$

$\bar{n} \circ \overline{P_0P} = 0 \Rightarrow$

$(a,b,c) \cdot (x - x_0,\ y - y_0,\ z - z_0) = 0 \Rightarrow$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0} \Rightarrow$$

$$\boxed{ax + by + cz + d = 0} \qquad d = -ax_0 - by_0 - cz_0$$

Eksempel 16

Bestem ligningen for y-z planen

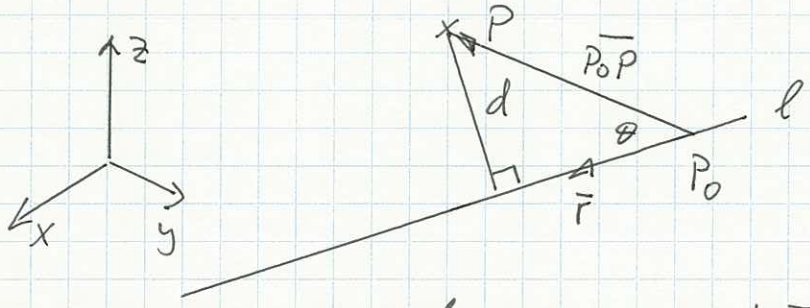
$\bar{n} = (1,0,0) \qquad P_0 = (0,0,0)$

$1(x - 0) + 0(y - 0) + 0(z - 0) = 0 \Rightarrow$

$$\underline{\underline{x = 0}}$$

Afstande i rummet

Afstand imellem punkt og linje



$\bar{r} \parallel l$

$P_0 \in l$

$$\sin\theta = \frac{d}{|\overline{P_0P}|} \Rightarrow d = |\overline{P_0P}| \sin\theta \Rightarrow$$

$$d = \frac{|\bar{r}||\overline{P_0P}| \sin\theta}{|\bar{r}|} \Rightarrow \boxed{d = \frac{|\bar{r} \times \overline{P_0P}|}{|\bar{r}|}}$$

Eksempel 17

Beregn afstanden mellem $P = (4, -1, 6)$ og linjen $l: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix}$ $\wedge t \in \mathbb{R}$

$\overline{P_0P} = (4-1, -1-2, 6-3) = (3, -3, 3)$

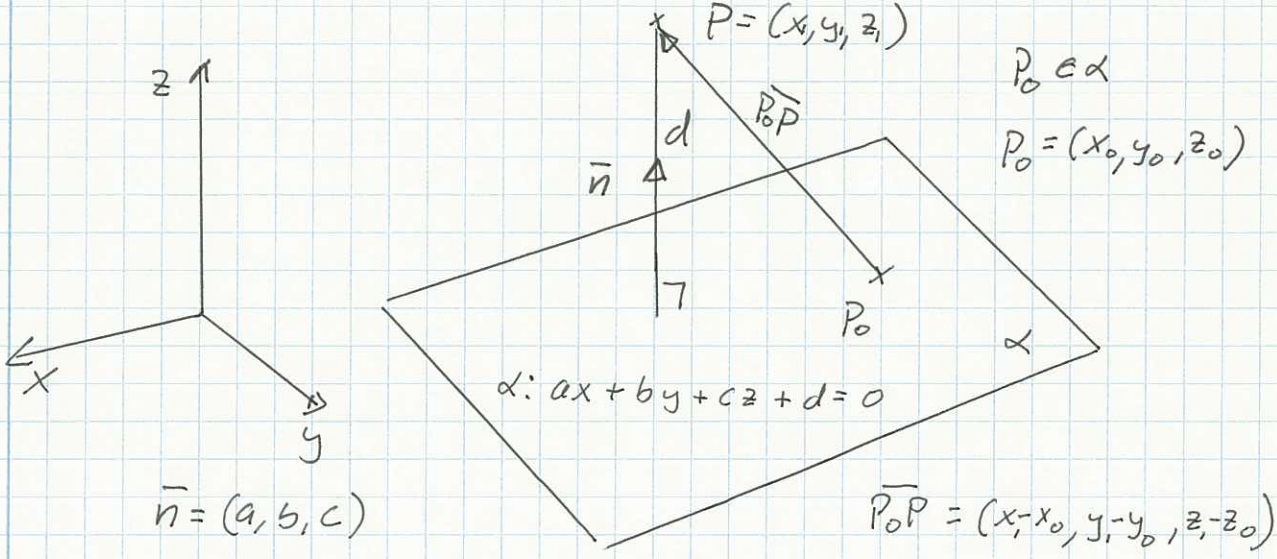
$$\bar{r} \times \overline{P_0P} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 5 & 8 & 9 \\ 3 & -3 & 3 \end{vmatrix} = \left(\begin{vmatrix} 8 & 9 \\ -3 & 3 \end{vmatrix} \; \begin{vmatrix} 9 & 5 \\ 3 & 3 \end{vmatrix} \; \begin{vmatrix} 5 & 8 \\ 3 & -3 \end{vmatrix} \right) = (51, 12, -39)$$

$$\left| \bar{r} \times \overline{P_0P} \right| = \sqrt{51^2 + 12^2 + 39^2} \approx 65{,}31$$

$$|\bar{r}| = \sqrt{25 + 64 + 81} \approx 13{,}04$$

$$d = \frac{65{,}31}{13{,}04} \approx \underline{\underline{5{,}01}}$$

Afstand imellem punkt og plan



$\bar{n} = (a, b, c)$

$\bar{n} \perp \alpha$

$P_0\bar{P} = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$

$$d = \left| \overline{P_0P_n} \right| = \left| \frac{\bar{n} \cdot \overline{P_0P}}{|\bar{n}|^2} \cdot \bar{n} \right| = \frac{|\bar{n} \cdot \overline{P_0P}|\,|\bar{n}|}{|\bar{n}|^2} \Rightarrow$$

$$d = \frac{|\bar{n} \cdot \overline{P_0P}|}{|\bar{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \Rightarrow$$

$$\boxed{d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}}$$

Eksempel 18

Beregn afstanden mellem $P = (1, 2, -1)$ og α: $3x - 2y + z - 1 = 0$

$$d = \frac{|3 - 4 - 1 - 1|}{\sqrt{9 + 4 + 1}} = \frac{3}{\sqrt{14}} \approx \underline{\underline{0{,}802}}$$

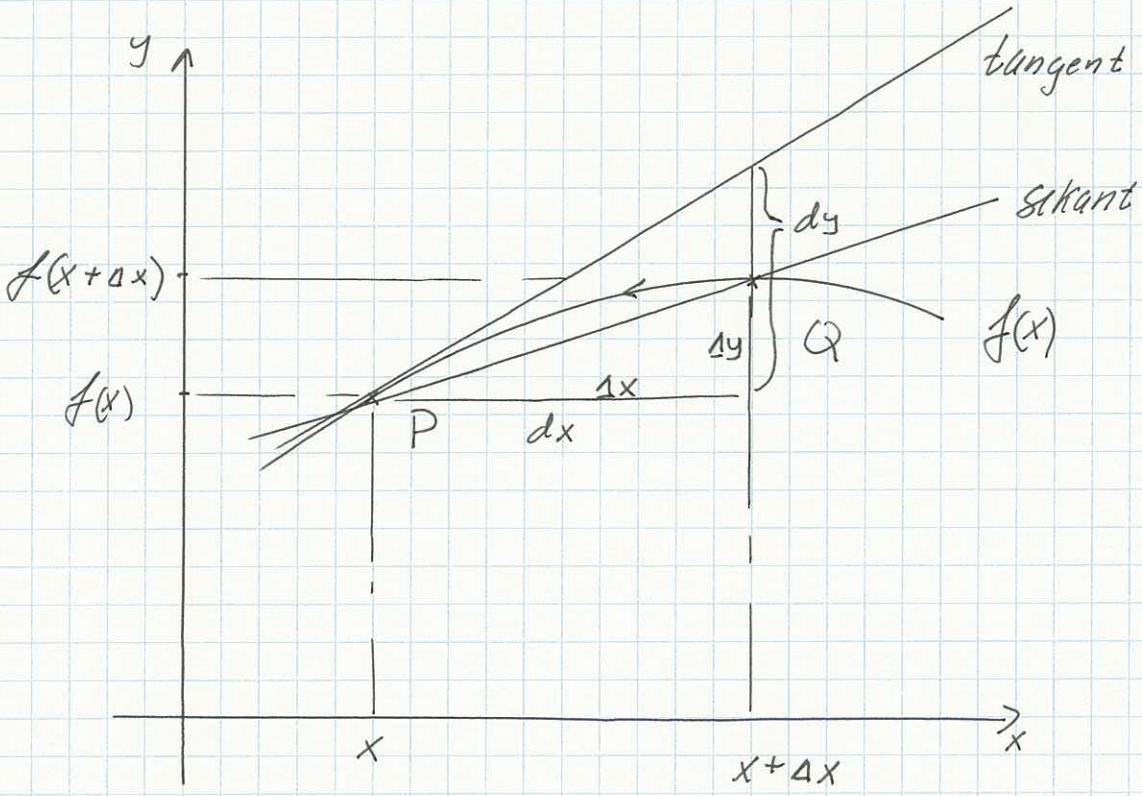
Eksempel 19

Beregn afstanden mellem origo og

α: $3x - 2y + z - 1 = 0$

$$d = \frac{|-1|}{\sqrt{9 + 4 + 1}} = \frac{1}{\sqrt{14}} \approx \underline{\underline{0{,}267}}$$

Differentialregning



Hældningskoefficienten til sekanten igennem P og Q er α_s

$$\alpha_s = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Hældningskoefficienten til tangenten igennem P er α_t

$$\boxed{\alpha_t = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{dy}{dx}}$$

$\frac{dy}{dx}$ kaldes differentialkvotienten

Den første afledede funktion

Hvis $y = f(x)$ er $\frac{dy}{dx} = f'(x)$ og

$f'(x)$ kaldes den første afledede funktion

$f''(x)$ kaldes den anden afledede funktion

etc.

$$y = f(x) = x^n \Rightarrow$$

$$y' = f'(x) = \frac{dy}{dx} = n x^{n-1} \Rightarrow$$

$$dy = n x^{n-1} dx \Rightarrow$$

$$dx^n = n x^{n-1} dx$$

dy kaldes differentialet af y

dx^n er sædvanligvis differentialet af x^n

Tabel

Funktion	Differential	Differentialkvotient
$y = f(x)$	$dy = d f(x)$	$\frac{dy}{dx} = f'(x)$
x^n	$n x^{n-1} dx$	$n x^{n-1}$
e^x	$e^x dx$	e^x
a^x	$\ln a \, a^x dx$	$\ln a \, a^x$
$\ln x \wedge x > 0$	$\frac{1}{x} dx$	$\frac{1}{x}$
$\log x \wedge x > 0$	$\frac{1}{\ln 10} \cdot \frac{1}{x} dx$	$\frac{1}{\ln 10} \cdot \frac{1}{x}$
$\sin x$	$\cos x \, dx$	$\cos x$
$\cos x$	$-\sin x \, dx$	$-\sin x$
$\tan x \wedge x \neq \frac{\pi}{2} + p\pi$	$(1 + \tan^2 x) dx$	$1 + \tan^2 x = \frac{1}{\cos^2 x}$

Regneregler for differentiation

Funktion		Differentialkvotient
sum	$f(x)+g(x)$	$f'(x)+g'(x)$
differens	$f(x)-g(x)$	$f'(x)-g'(x)$
multiplikation med en konstant	$c\cdot f(x)$	$c\cdot f'(x)$
produkt	$f(x)\cdot g(x)$	$f'(x)\,g(x)+f(x)\,g'(x)$
kvotient $\wedge\ g(x)\neq 0$	$\frac{f(x)}{g(x)}$	$\frac{f'(x)\,g(x)-f(x)\,g'(x)}{(g(x))^2}$
konstant	K	0

Eksempel 20

Givet $f(x) = e^x \cos x$

Bestem $f'(x)$

$$f'(x) = e^x \cos x - e^x \sin x \Rightarrow$$
$$\underline{\underline{f'(x) = e^x(\cos x - \sin x)}}$$

Eksempel 21

Givet $f(x) = \frac{\sin x}{\cos x}$

Bestem $f'(x)$

$$f'(x) = \frac{\cos x \cos x + \sin x \sin x}{\cos^2 x} = \underline{\underline{\frac{1}{\cos^2 x}}}$$

men det var jo ogsa° differentialkvotienten til $f(x) = \tan x$

Differentiation af sammensatte funktioner

$$y = g(f(x)) \Rightarrow \boxed{y' = g'(f(x)) \cdot f'(x)}$$

Eksempel 22

Givet $f(x) = e^{\sin 3x}$

Bestem $f'(x)$

$$f'(x) = e^{\sin 3x} \cdot \cos 3x \cdot 3 \Rightarrow$$

$$\underline{\underline{f'(x) = 3e^{\sin 3x} \cdot \cos 3x}}$$

Eksempel 23

Givet $f(x) = \dfrac{\ln \cos x}{e^{\tan x}}$

$$f'(x) = \frac{\frac{1}{\cos x}(-\sin x) \cdot e^{\tan x} - \ln \cos x \cdot e^{\tan x} \cdot \frac{1}{\cos^2 x}}{\left(e^{\tan x}\right)^2}$$

$$\Rightarrow f'(x) = \frac{-\tan x \cdot e^{\tan x} - \frac{\ln \cos x}{\cos^2 x} \cdot e^{\tan x}}{e^{2\tan x}} \Rightarrow$$

$$f'(x) = \frac{e^{\tan x}\left(-\tan x - \frac{\ln \cos x}{\cos^2 x}\right)}{e^{2\tan x}} \Rightarrow$$

$$f'(x) = \frac{-\tan x - \frac{\ln \cos x}{\cos^2 x}}{e^{\tan x}} \Rightarrow$$

$$\underline{\underline{f'(x) = \frac{-\tan x \cos^2 x - \ln \cos x}{e^{\tan x} \cdot \cos^2 x}}}$$

Integralregning

Stamfunktioner

Hvis $F(x)$ er stamfunktion til $f(x)$, så er $F'(x) = f(x)$

Det betyder også at $F(x) + C$ er stamfunktion til $f(x)$ idet

$$(F(x) + C)' = F'(x) + C' = F'(x) + 0 = F'(x) = f(x)$$

$$F'(x) = f(x) \Rightarrow \frac{dF(x)}{dx} = f(x) \Rightarrow$$

$$dF(x) = f(x)\,dx \Rightarrow$$

$$\int dF(x) = \int f(x)\,dx \Rightarrow$$

$$\boxed{F(x) = \int f(x)\,dx}$$

Stamfunktionen er lig med <u>det ubestemte integral</u> af $f(x)$

Integrationsprøven

$$\boxed{\left(F(x)\right)' = \left(\int f(x)\,dx\right)' \Rightarrow F'(x) = f(x)}$$

Integral- og differentialregning er modsatte regningsarter.

Det ubestemte integral af en funktion er altså et uendeligt antal af funktioner, der kun adskilles af konstanter.

Graferne for funktionerne $F(x)+C_1$, $F(x)+C_2$, $F(x)+C_3$ etc. kaldes <u>integralkurver</u>.

Tabel over funktioner og deres stamfunktioner

$f(x)$	$F(x)$
x^n	$\frac{x^{n+1}}{n+1}+C \wedge n \neq -1$
$x^{-1}=\frac{1}{x}$	$\ln x + C \wedge x>0$
e^x	$e^x + C$
a^x	$\frac{1}{\ln a}\, a^x + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$1+\tan^2 x$	$\tan x + C$
$\ln x \quad \wedge x>0$	$x\ln x - x + C$
$\log x \quad \wedge x>0$	$x\log x - \frac{x}{\ln 10} + C$
$\tan x \quad \wedge x \neq \frac{\pi}{2}+p\pi$	$-\ln\cos x + C$

Regneregler for ubestemte integraler

$$\int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int (f(x)-g(x))\,dx = \int f(x)\,dx - \int g(x)\,dx$$

$$\int c\,f(x)\,dx = c\int f(x)\,dx$$

Eksempel 24

Beregn $\int (3e^x - \sin x + x^4)\,dx$

$$\int (3e^x - \sin x + x^4)\,dx = 3\int e^x dx - \int \sin x\,dx + \int x^4 dx$$

$$= \underline{\underline{3e^x + \cos x + \tfrac{1}{5}x^5 + C}}$$

Integrationsprøven

$$\left(3e^x + \cos x + \tfrac{1}{5}x^5 + C\right)' = 3e^x - \sin x + x^4$$

Integration ved substitution

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(t)\,dt = F(t) + k =$$

$$F(g(x)) + k \qquad t = g(x) \Rightarrow dt = g'(x)\,dx$$

Eksempel 25

Beregn $\int \tan x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx$$

$t = \cos x \Rightarrow dt = -\sin x\,dx$

$$-\int \frac{-\sin x}{\cos x}\,dx = -\int \frac{dt}{t} = -\ln t + C =$$

$$\underline{\underline{-\ln \cos x + C}}$$

Ofte er det nemmere, at anvende den viden man har om differentialer.

$d\cos x = -\sin x\,dx$

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x} = -\int \frac{d\cos x}{\cos x} = \underline{\underline{-\ln \cos x + C}}$$

Eksempel 26

Bergn $\int 4x\sqrt{x^2+4}\,dx$

$d(x^2+4) = 2x\,dx$

$\int 4x\sqrt{x^2+4}\,dx = 2\int 2x\sqrt{x^2+4}\,dx =$

$2\int (x^2+4)^{\frac{1}{2}}\,d(x^2+4) = 2\,(x^2+4)^{\frac{3}{2}}\cdot\frac{2}{3} + C =$

$\underline{\underline{\frac{4}{3}(x^2+4)^{\frac{3}{2}} + C}}$

Eksempel 27

Bergn $\int x^2\cdot e^{x^3}\,dx$

$d\,x^3 = 3x^2\,dx$

$\int x^2 e^{x^3}\,dx = \frac{1}{3}\int 3x^2 e^{x^3}\,dx = \frac{1}{3}\int e^{x^3}\,dx^3$

$= \underline{\underline{\frac{1}{3}e^{x^3} + C}}$

Eksempel 28

Bergn $\int e^{-2x}\,dx$

$d(-2x) = -2\,dx$

$\int e^{-2x}\,dx = -\frac{1}{2}\int -2e^{-2x}\,dx = -\frac{1}{2}\int e^{-2x}\,d(-2x)$

$= \underline{\underline{-\frac{1}{2}e^{-2x} + C}}$

Intigrationsprøven $\left(-\frac{1}{2}e^{-2x} + C\right)' =$

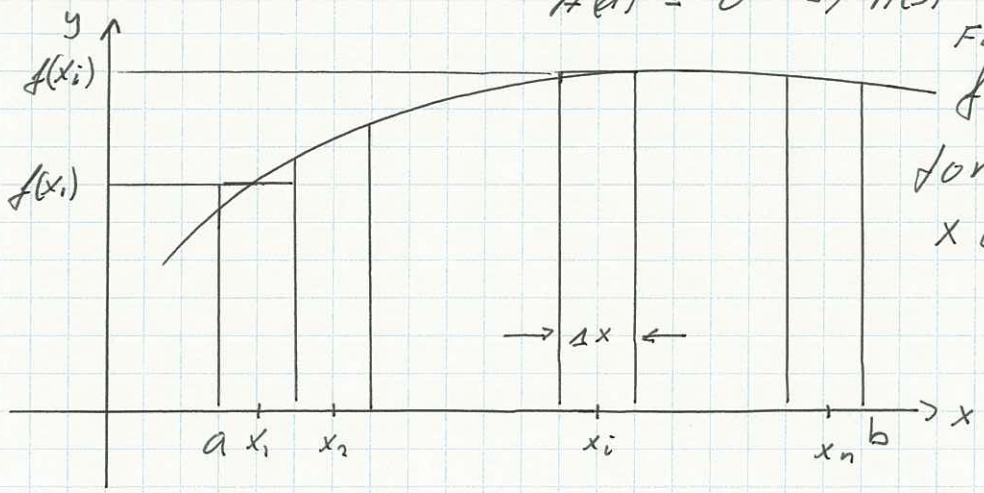
$-\frac{1}{2}e^{-2x}(-2) = e^{-2x}$

Det bestemte integral

Forudsætninger:

Arealfunktionen $A(x) = F(x) + K$

$A(a) = 0 \Rightarrow A(b) = A$



Forudsætning:

$f(x) > 0$ for alle $x \in [a, b]$

$A_i \cong f(x_i)\,\Delta x$

$A \cong f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_i)\,\Delta x + \cdots + f(x_n)\,\Delta x$

$\Rightarrow A \cong \sum_{i=1}^{n} f(x_i)\,\Delta x$ middelsummen

$$\boxed{A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx}$$

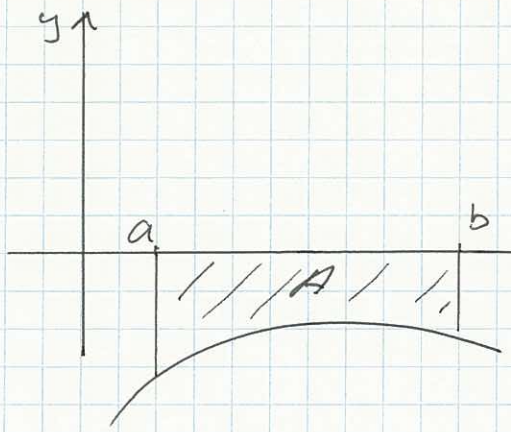
$\int_a^b f(x)\,dx$ kaldes det bestemte integral af $f(x)$ fra a til b

Fundamentalsætningen

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Det bestemte integral er et tal eksempelvis et areal eller et volumen

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$



$$\int_a^b f(x)\,dx = F(b) - F(a) < 0$$

$$A = -\int_a^b f(x)\,dx = -F(b) + F(a) =$$

$$F(a) - F(b) = \int_b^a f(x)\,dx > 0$$

Regneregler for bestemt integralet

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt = \left[F(t)\right]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

Eksempel 29

Beregn $\int_0^{\pi} \sin x \, dx$

$$\int_0^{\pi} \sin x \, dx = -\cos x \Big]_0^{\pi} = -\cos \pi + \cos 0 = \underline{\underline{2}}$$

Eksempel 30

Beregn $\int_0^{2\pi} \sin x \, dx$

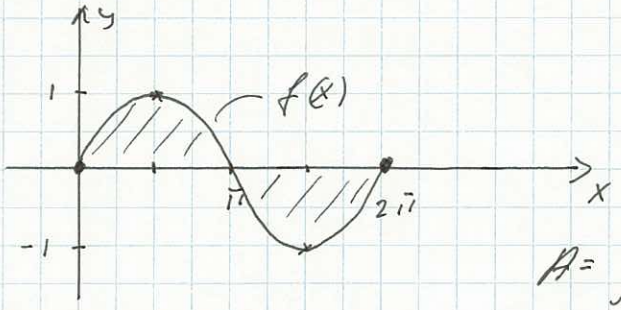
$$\int_0^{2\pi} \sin x \, dx = -\cos x \Big]_0^{2\pi} = -\cos 2\pi + \cos 0 = \underline{\underline{0}}$$

Eksempel 31

Givet funktionen $f(x) = \sin x \quad \wedge \quad 0 \leq x \leq 2\pi$

a/ Tegn funktionen

b/ Beregn arealet der afgrænses af $f(x)$ og x-aksen.



$$A = \int_0^{\pi} f(x) \, dx - \int_{\pi}^{2\pi} f(x) \, dx \Rightarrow$$

$$A = 2 \int_0^{\pi} \sin x \, dx = 2 \cdot 2 = \underline{\underline{4}}$$

Eksempel 32

Bestem volumenet af en kugle med radius 3

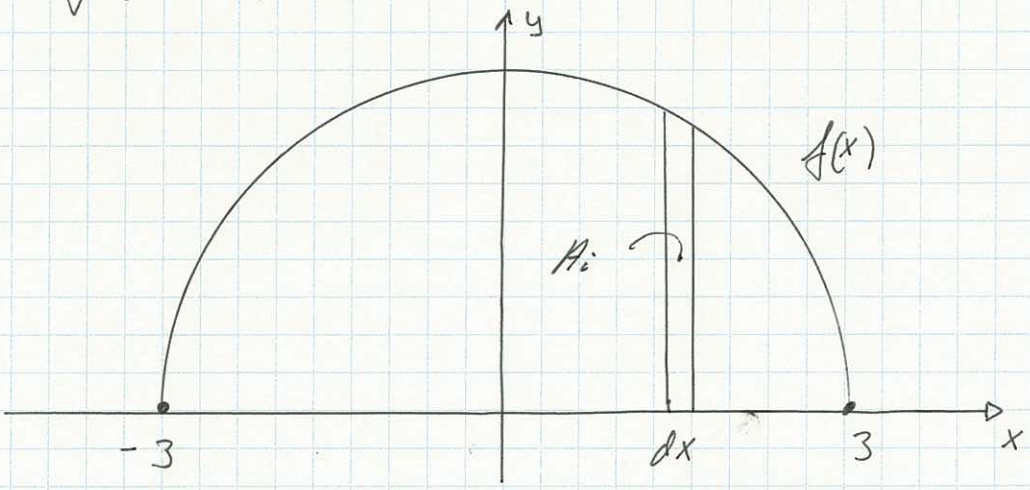
$x^2 + y^2 = r^2$ kurven for en cirkel med centrum $C = (0,0)$ og radius r

$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

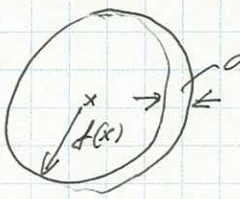
$y \geq 0 \Rightarrow y = \sqrt{r^2 - x^2} \wedge -r \leq x \leq r$

Vi kan kun dreje en funktion for en halvcirkel.

$f(x) = \sqrt{9 - x^2} \wedge -3 \leq x \leq 3$



Ai drejes 360° omkring x-aksen



$dV_x = \pi f(x)^2 dx$

$V_{kugle} = 2 \int_0^3 \pi f(x)^2 dx = 2\pi \int_0^3 (9 - x^2) dx$

$= 2\pi \left[9x - \frac{1}{3}x^3 \right]_0^3 = 2\pi (27 - 9) = \underline{\underline{36\pi}}$

Sammenlign med formlen

$V_k = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot 27 = 36\pi$